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Chaos Maximizing Optimal Control

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A chaos maximizing optimal control problem is formulated and applied to Duffing's equation to maximize the largest Lyapunov exponent. The resulting bang-bang optimal controller yields a positive value of the largest Lyapunov exponent, indicating chaotic behavior. In fact, the largest Lyapunov exponent is approximately twice as large as that achieved with simple sinusoidal forcing at the same amplitude bounds. However, the resulting phase portrait of the optimal trajectory is a limit cycle and is not chaotic at all. This paradoxical result contradicts the basic theory that a bounded trajectory with at least one positive Lyapunov exponent must be chaotic. Details concerning the development of a chaos measurement that is viable for current optimal control theory, a method of continuous normalization, the paradoxical chaotic limit cycle, resolution of the paradox, and closed-loop optimal jump condition in an augmented space are presented. In particular, for systems of differential equations with only piecewise differentiable right-hand sides due to a switching control, a jump discontinuity condition must be imposed on the state perturbations in order to compute correct Lyapunov exponents.

Key Words: Chaos, Maximum Chaos, Optimal Control, Strange Attractor, Limit Cycle Continuous Normalization, Perturbation Vector, Jump Condition.

1. Introduction

The report of "strange" behavior in Lorenz's weather prediction model (Lorenz, 1963) and the existence of modern computing systems triggered the study of chaos by many scientists and mathematicians. Recently, researchers have reported that "strange" behaviors are found not only in the dynamics field, including discrete-time systems such as a prey-predator model, but also in systems involving feedback controls. Many cases have been reported in which chaos exists in either practical control systems or numerical computer models (Brockett, 1982; Ushio, 1983; Rubio, 1985; Cook, 1985; Cook, 1986). Most of the systems considered as examples were simple systems of dimension less than two and the controls were also elementary, such as linear constant-gain feedback or switching functions with hysteresis. Furthermore, none of the reports mentioned chaos found in an optimal control system or even the possibility of the existence of chaos in a nonlinear optimal control system.

In this paper, a study of the chaotic responses of nonlinear dynamical systems subject to optimal control is presented. Three difficulties has been considered. The first is the current lack of a chaos measurement suitable for the optimal control theory. Several measurements have been developed including Lyapunov exponents and various measures of fractal dimension (Capacity dimension, Hausdorff dimension, Information dimension, Pointwise dimension, and Lyapunov dimension). However the chaos measure should be expressed in integral form, with corresponding differential equation, so that standard optimal control theory (Pontryagin, 1964) can be used. We will employ the largest Lyapunov exponent for a chaos measurement since the formula for the largest Lyapunov exponent can be converted to integral form. The second is the exponential growth of both state perturbations and optimal control adjoint variables, leading to over flow

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error in the digital computer during numerical simulation. The third difficulty, which is related to the overflow problem, is that the current reorthonormalization procedure for the calculation of Lyapunov exponents induces discontinuous jump changes in the variables of the state perturbation differential equations (Wolf, 1985). One fundamental technique is to replace periodic discontinuous renormalization with differential equations that correspond to continuous normalization at each time instant.

In Section 2 and 3, A Lyapunov exponent as a performance index for a chaos extremizing control problem is considered. To be a proper performance index, the object function must be in an integral form. A technique that converts the first Lyapunov exponent into a differential equation and further into an integral form is introduced. In Section 4, for the Duffing's Oscillator, it has been shown that maximizing the first Lyapunov exponent is equivalent to maximizing the Lyapunov dimension conjectured by Kaplan and Yorke. This result enables us to maximize chaos by indirectly manipulating a Lyapunov exponent for this particular system.

In the following sections, a numerical simulation has been conducted and a paradoxical result called "Chaotic Limit Cycle" is observed. The simulation result is paradoxical since the state space trajectory is a limit cycle and one of the Lyapunov exponents is a positive number. This result must be wrong since the basic understanding that a bound trajectory with at least one positive exponent must be chaotic. To resolve this paradox, a state perturbation vector jump condition has been developed. However, the jump condition developed in the augmented space requires prior knowledge of the optimal cost return function.

Finally, in Section 10, the necessary jump condition has been applied to a system that we know the cost return function. It is also shown graphically that the perturbation vector crossing a discontinuity manifold exhibits correct result, in the sense of Lyapunov exponent computation, with the jump condition implemented.

In particular, we propose a necessary jump condition that should be applied when the Lyapunov exponent is used as a performance index for chaos extremizing control problems. Note that one must know the optimal cost return function for the optimal problem to compute correct initial values of the perturbation vector.

2. Lyapunov Exponents As a Performance Index

To discuss various topics related to a chaotic system, consider a control system of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{1}$$

assuming that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is continuous and differentiable for all $\boldsymbol{x}$ and $\boldsymbol{u}$. The Lyapunov exponents of the continuous-time dynamical system (1) are defined by letting $\delta(t)$ denote the distance between two trajectories for a continuous-time dynamical system. If there exists a number σ which satisfies

$$\delta(t) \rightarrow \delta(0)e^{\sigma t} \text{ as } t \rightarrow \infty \tag{2}$$

for arbitrary small $\delta(0)$, then σ is called a Lyapunov exponent and is defined by

$$\sigma = \lim_{t \to \infty}\left(\lim_{\delta(0) \to 0} \frac{1}{t} ln\left(\frac{\delta(t)}{\delta(0)}\right)\right) \tag{3}$$

Let $\boldsymbol{x}(t) \triangleq \phi(t, \boldsymbol{x}(0))$ be a solution generated by (1) with admissible control $\boldsymbol{u}$. To compute the Lyapunov exponents, consider an n-dimensional ellipsoid whose center lies on the reference trajectory $\boldsymbol{x}$ and semi-axes are determined by n-orthogonal perturbation vectors. Let $\boldsymbol{z}(t) \triangleq \phi(t, \boldsymbol{z}(0))$ be the solution of (1) generated by the same function $\boldsymbol{u}(\cdot)$ with the initial value $\boldsymbol{z}(0)$ being arbitrarily close to $\boldsymbol{x}(0)$.

Applying Taylor's theorem, assuming the right hand side of (1) is continuous and continuously differentiable along $\boldsymbol{x}(t)$, we have

$$\boldsymbol{z}(t) = \boldsymbol{x}(t) + \varepsilon\boldsymbol{\eta}(t) + \boldsymbol{O}(\varepsilon), \tag{4}$$

where $\boldsymbol{O}(\varepsilon)/\varepsilon \rightarrow \boldsymbol{0}$ and $\boldsymbol{\eta}(t) \in E^n$ satisfies the linearized state perturbation equations

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{D}(t)\boldsymbol{\eta} \\ \mathbf{D}(t) &= \left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right) \end{aligned} \tag{5}$$

along $\boldsymbol{x}(t)$ using $\boldsymbol{u}(\cdot)$. Replacing $\boldsymbol{\delta}(0)$ with z

(0)-$\boldsymbol{x}(0)$ and $\boldsymbol{\delta}(t)$ with $z(t)$-$\boldsymbol{x}(t)$ in Eq. (3), we have

$$\sigma=\lim_{t\to\infty}\frac{1}{t}\left(\frac{\|\boldsymbol{\eta}(t)\|}{\|\boldsymbol{\eta}(0)\|}\right) \tag{6}$$

as $\varepsilon\to 0$. Note that Eq. (6) handles the limiting process for $\boldsymbol{\delta}(0)\to 0$ in (3) automatically.

By the definition of the Lyapunov exponents, semi-axes grow exponentially with time and diverge in magnitude beyond the capacity that a finite-word-length computer can handle. This is not a mathematical problem but a computational problem (Wolf, 1985). Another peculiarity (both mathematical and computational) is that each perturbation vector has a tendency, over time, to align itself along the direction corresponding to the largest Lyapunov exponent. The first problem may be circumvented by renormalization of the perturbation vectors when their magnitudes become big. The second problem can be solved by repeated use of reorthogonalization on the perturbation vectors. A method of Gramm-Schmidt reorthonormalization is presented in the reference (Wolf, 1985).

An optimal control problem maximizing the chaos of the system by manipulating the first Lyapunov exponent, described by the admissible rules of the form (1), can be formulated using the quantity σ defined in (6) as an performance index. To convert Eq. (6) to integral form, we differentiate σ in (6) with respect to t :

$$\begin{aligned}\dot{\sigma} &= \frac{d}{dt}\left[\frac{1}{t}ln\frac{\|\boldsymbol{\eta}\|}{\|\boldsymbol{\eta}_0\|}\right]\\ &= \frac{1}{t}\frac{\boldsymbol{\eta}^T\dot{\boldsymbol{\eta}}}{\|\boldsymbol{\eta}\|^2}-\frac{1}{t^2}ln\frac{\|\boldsymbol{\eta}\|}{\|\boldsymbol{\eta}_0\|}\\ &= \frac{1}{t}\frac{\boldsymbol{\eta}^T\dot{\boldsymbol{\eta}}}{\|\boldsymbol{\eta}\|^2}-\frac{\sigma}{t}\end{aligned} \tag{7}$$

provided that $t\neq 0$. Finally we have an optimal control problem, together with (1), (5) and (7), as follows :

$$\max_{u}\int_{t_0}^{t_f}\dot{\sigma}\,dt$$

subject to

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x},\ \boldsymbol{u})\\ \dot{\boldsymbol{\eta}} &= \mathbf{D}(t)\boldsymbol{\eta}\\ \dot{\sigma} &= \frac{1}{t}\frac{\boldsymbol{\eta}^T\dot{\boldsymbol{\eta}}}{\|\boldsymbol{\eta}\|^2}-\frac{\sigma}{t}.\end{aligned} \tag{8}$$

and $\boldsymbol{x}(t_0)$, $\boldsymbol{\eta}(t_0)$, t_0 and t_f are given.

3. Continuous Normalization

As discussed earlier, state perturbations grow exponentially. To remedy this problem, one may renormalize state perturbation vectors to unit vectors periodically (Wolf, 1985). However, this renormalization procedure for the computation of Lyapunov exponents causes discontinuities in the variables of the state perturbations. In this paper, we employ a method of "continuous normalization" which replaces periodic discontinuous renormalization with differential equations that correspond to continuous normalization at each time instance. This technique has been developed by Lee, Grantham, and Fisher (1994) and the relevant part of their work to this research is repeated here.

Let $\xi\triangleq r(\boldsymbol{\eta}/\|\boldsymbol{\eta}\|)$be a normalized perturbation vector and a constant r be the norm of ξ. Differentiating ξ with respect to time, we get

$$\begin{aligned}\dot{\boldsymbol{\xi}} &= r\left(\frac{\dot{\boldsymbol{\eta}}}{\|\boldsymbol{\eta}\|}-\frac{\boldsymbol{\eta}\boldsymbol{\eta}^T\dot{\boldsymbol{\eta}}}{\|\boldsymbol{\eta}\|^3}\right)\\ &= \frac{r}{\|\boldsymbol{\eta}\|}\left(\boldsymbol{I}-\frac{\boldsymbol{\eta}\boldsymbol{\eta}^T}{\|\boldsymbol{\eta}\|^2}\right).\end{aligned} \tag{9}$$

Substituting $\dot{\boldsymbol{\eta}}$ from (5) into (9), we obtain

$$\dot{\boldsymbol{\xi}}=\left(\boldsymbol{I}-\frac{\boldsymbol{\xi}\boldsymbol{\xi}^T}{r^2}\right)\mathbf{D}(t)\boldsymbol{\xi} \tag{10}$$

and corresponding $\dot{\sigma}$ in (8)

$$\dot{\sigma}=\frac{1}{r^2t}\boldsymbol{\xi}^T\mathbf{D}(t)\boldsymbol{\xi}-\frac{\sigma}{t} \tag{11}$$

provided that $t\neq 0$. Advantages of this approach are the magnitude of the perturbation vector $\boldsymbol{\xi}$ stays constant (we have not yet proved the stability of Eq. (11), but it turned out to be stable numerically) and there is no discontinuity in the state perturbation vector.

It is still undesirable to have t in the denominator of Eq. (11) which prevents choosing $t_0=0$. Moving the term σ/t in (11) to the left and multiplying both sides by t, we have

$$t\left(\dot{\sigma}+\frac{\sigma}{t}\right)=\frac{d}{dt}(t\sigma)=\frac{\boldsymbol{\xi}^T\mathbf{D}(t)\boldsymbol{\xi}}{r^2} \tag{12}$$

With an arbitrary choice of t_0 and $t_f\neq 0$, we finally have

$$\sigma(t_f)=\frac{1}{t}\int_{t_0}^{t_f}\frac{\boldsymbol{\xi}^T\mathbf{D}(t)\boldsymbol{\xi}}{r^2}dt+\frac{\sigma(t_0)t_0}{t_f}. \quad (13)$$

Note that we may choose $t_0=0$ or $\sigma(t_0)=0$ to nullify the right most term of (13). However, it will vanish as $t_f\to\infty$ so that we can ignore it for Lyapunov exponent computation purposes.

Using (13) as a performance index, the optimal control problem for the maximum Lyapunov exponent is formulated as follows:

$$\max_{u}\int_{t_0}^{t_f} f_0 dt$$

subject to

$$\begin{aligned}
&\dot{\boldsymbol{x}}_0=f_0\triangleq\frac{\boldsymbol{\xi}^t\mathbf{D}(t)\boldsymbol{\xi}}{r^2}\\
&\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})\\
&\dot{\boldsymbol{\xi}}=\left(\boldsymbol{I}-\frac{\boldsymbol{\xi}\boldsymbol{\xi}^T}{r^2}\right)\mathbf{D}(t)\boldsymbol{\xi}\\
&\boldsymbol{u}\in\mathbf{U},\ t_0\le t\le t_f
\end{aligned} \quad (14)$$

where $\boldsymbol{x}(t_0)$, $\boldsymbol{\xi}(t_0)$, t_0 and t_f are given, $\mathbf{U}$ is a control constraint set, and $r=\|\boldsymbol{\xi}\|$. Eq. (14) constitute a set of $2n+1$ differential equations whose solutions define trajectories in E^{2n+1}. In order to utilize the maximum principle (Pontryagin, 1964), we require the H function given by

$$\boldsymbol{H}=\lambda_0 f+\boldsymbol{\lambda}_x^T\boldsymbol{f}+\boldsymbol{\lambda}_\xi^T\dot{\boldsymbol{\xi}} \quad (15)$$

and differential equations for corresponding adjoint vectors

$$\dot{\boldsymbol{\lambda}}_x^T=-\frac{\partial H}{\partial \boldsymbol{x}} \quad (16)$$

$$\dot{\boldsymbol{\lambda}}_\xi^T=-\frac{\partial H}{\partial \boldsymbol{\xi}}. \quad (17)$$

The optimal control $\boldsymbol{u}^*$ is chosen to satisfy following conditions given by

$$\begin{aligned}
&\sup_{\boldsymbol{u}\in \boldsymbol{U}} H(\boldsymbol{\lambda}(t),\boldsymbol{x}^*(t),\boldsymbol{\xi}^*(t),\boldsymbol{u})\\
&=H(\boldsymbol{\lambda}(t),\boldsymbol{x}^*(t),\boldsymbol{\xi}^*(t),\boldsymbol{u}^*(t)),\\
&H(\boldsymbol{\lambda}(t),\boldsymbol{x}^*(t),\boldsymbol{\xi}^*(t),\boldsymbol{u}^*(t))=0\\
&\forall t\in[t_0,t_f]
\end{aligned} \quad (18)$$

where

$$\boldsymbol{\lambda}=\begin{Bmatrix}\lambda_0\\ \boldsymbol{\lambda}_x\\ \boldsymbol{\lambda}_\xi\end{Bmatrix}.$$

4. Duffing's Oscillator

The equation chosen for the study is

$$\ddot{y}+\delta\dot{y}-\frac{1}{2}(y-y^3)=u(t) \quad (19)$$

or, in a state space form,

$$\begin{aligned}
&\dot{x}_1=x_2\\
&\dot{x}_2=\frac{1}{2}(x_1-x_1^3)-\delta x_2+u(t)
\end{aligned} \quad (20)$$

where δ is a fixed parameter and u acts as a control variable. Given the system (19) or (20), one may use the optimal control problem setup in (14) to seek maximum chaos.

We can maximize the dimension of a chaotic attractor for the example system via indirect manipulation of Lyapunov exponents, especially the first one, based on the conjecture of Kaplan and Yorke (1979). Since the order of the system is three and it is required that, in order to be chaotic for third-order dissipative systems, the first, second, and third Lyapunov exponents are positive, zero and negative, respectively (Haken, 1983). So the Lyapunov dimension of Kaplan and Yorke is

$$d_L=2+\frac{\sigma_1}{|\sigma_3|} \quad (21)$$

and the sum of the Lyapunov exponents is equal to the divergence which is a negative constant number. In a mathematical form, the divergence is

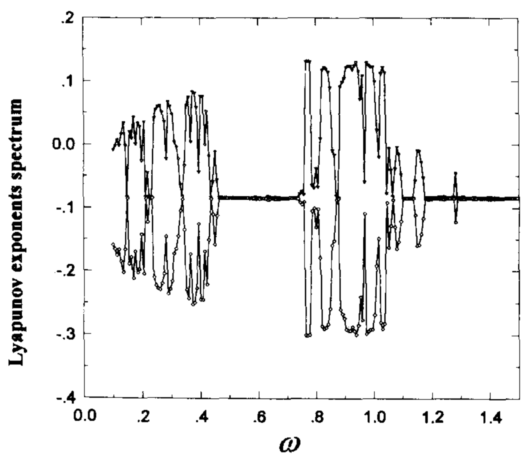

Fig. 1 Lyapunov exponents spectrum for Duffing's equation with simple sinusoidal forcing

given by

$$\nabla\cdot \boldsymbol{f}=\sigma_1+\sigma_2+\sigma_3=-\delta. \tag{22}$$

From (21) and (22), we see that the Lyapunov dimension is a monotonic function of σ_1 as follows:

$$d_L=2+\frac{\sigma_1}{|\sigma_1+\delta|}. \tag{23}$$

Thus, we maximize the first Lyapunov exponent indirectly instead of manipulating the dimension of an attractor to maximize chaos of the system.

In the study we will use $\delta=0.168$ in Eq. (20). The Lyapunov exponents spectrum of the system with a simple sinusoidal excitation having the same magnitude as the optimal control is plotted in Fig. 1.

5. Optimal Control Problem Incorporating Continuous Renormalization and Normalized Adjoint Vector

The magnitude of some of the adjoint vector components grows exponentially as does the magnitude of state perturbation vectors corresponding to positive Lyapunov exponents. To remedy this problem, we again introduce normalized adjoint variables and develop governing differential equations. To do so, let us introduce the renormalized adjoint variable vector and a new H function, $I\!H$, that consists of the velocity vector and the renormalized adjoint vector as follows:

$$\Lambda\triangleq\nu\frac{\boldsymbol{\lambda}}{\|\boldsymbol{\lambda}\|} \text{ and } I\!H\triangleq\nu\frac{\boldsymbol{\lambda}^T}{\|\boldsymbol{\lambda}\|}\boldsymbol{F}=\Lambda^T\boldsymbol{F} \tag{24}$$

where $\nu=\|\Lambda\|$. Differentiating Λ and substituting Eqs. (16) and (17) into it, we have a differential equation for the renormalized adjoint vector represented by

$$\dot{\Lambda}=-\left(I-\frac{\Lambda\Lambda^T}{\nu^2}\right)\left(\frac{\partial\boldsymbol{F}}{\partial\boldsymbol{X}}\right)^T\Lambda \tag{25}$$

where

$$\boldsymbol{X}\triangleq\begin{Bmatrix}x_0\\ \boldsymbol{x}\\ \boldsymbol{\xi}\end{Bmatrix},\ \boldsymbol{F}\triangleq\begin{Bmatrix}f_0\\ \boldsymbol{f}\\ \dot{\boldsymbol{\xi}}\end{Bmatrix}.$$

We note that in (24), Λ_0 corresponding to λ_0 is no longer constant. Incorporating (13) as a performance index for maximum chaos, renormalized state perturbation in (14), renormalized adjoint variable vector in (25), and function in (24), we obtain the final form of the optimization problem as follows:

$$\max_{u}\int_{t_0}^{t_f} f_0 dt \tag{26}$$

subject to

$$\dot{\boldsymbol{X}}=\boldsymbol{F}(\boldsymbol{X}(t),\ \boldsymbol{u}(t))$$
$$\boldsymbol{u}\in \mathrm{U},\ t_0\le t\le t_f \tag{27}$$

and (25) where $\boldsymbol{X}(t_0)$, t_0, and t_f are given and f_0 is defined in (14). The optimal control u^* is chosen by the following conditions, given by

$$\sup_{\boldsymbol{u}\in\mathrm{U}} I\!H(\Lambda(t),\ \boldsymbol{X}^*(t),\ \boldsymbol{u})$$
$$=I\!H\Lambda(t),\ \boldsymbol{X}^*(t),\ \boldsymbol{u}^*(t)) \tag{28}$$
$$I\!H(\Lambda(t),\ \boldsymbol{X}^*(t),\ \boldsymbol{u}^*(t))=0$$
$$\forall t\in[t_0,\ t_f]. \tag{29}$$

6. Numerical Study

Consider the Duffing system in (20), a system of differential equations for the normalized state perturbations

$$\dot{\boldsymbol{\xi}}=\left(I-\frac{\boldsymbol{\xi}\boldsymbol{\xi}^T}{r^2}\right)\mathrm{D}(t)\boldsymbol{\xi}$$
$$=\begin{Bmatrix}\xi_2-\xi_1^2\xi_2/r^2-\frac{1}{2}(1-3x_1^2)\xi_1^2\xi_2/r^2+\delta\xi_1\xi_2^2/r^2\\ -\xi_1\xi_2^2/r^2+\frac{1}{2}(1-3x_1^2)(1-\xi_2^2/r^2)\xi_1-\delta\xi_2(1-x_2^2/r^2)\end{Bmatrix}, \tag{30}$$

and a performance functional

$$f_0=\frac{1}{r^2}\boldsymbol{\xi}^T\mathrm{D}(t)\boldsymbol{\xi}$$
$$=\frac{1}{r^2}\left[\frac{1}{2}(1-3x_1^2)\xi_1\xi_2+\xi_1\xi_2-\delta\xi_2^2\right] \tag{31}$$

with a scalar control input $u(t)$, subject to the constraints

$$|\mathrm{u}|\le \mathrm{u}_{\max} \tag{32}$$

and parameter values $\delta=0.168$ and $u_{\max}=0.25$.

Using (31) for the performance index, we set out to determine an optimal control obtained by the conditions in (28) and (29) subject to the constraint (32). Since u appears linearly in ,an optimal control u^* is bang-bang control, with the possibility of a singular control, and is determined by

$$u^*=\begin{cases} u_{\max} & \text{if } \Sigma>0 \\ -u_{\max} & \text{if } \Sigma<0 \\ \text{singular control} & \text{if } \Sigma=0 \end{cases} \tag{33}$$

where

$$\begin{aligned} u^*= & \Lambda_0\left(\frac{1}{r^2}\left\{\frac{1}{2}(1-3x_1^2)\xi_1\xi_2+\xi_1\xi_2-\delta\xi_2^2\right\}\right) \\ & +\Lambda_{x1}x_2 \\ & +\Lambda_{x2}\left(\frac{1}{2}(x_1-x_1^3)-\delta x_2+u(t)\right) \\ & +\Lambda_{\xi 1}\left(\xi_2-\xi_1^2\xi_2/r^2-\frac{1}{2}(1-3x_1^2)\xi_1^2\xi_2/\right. \\ & \left. r^2+\delta\xi_1\xi_2^2/r^2\right) \\ & +\Lambda_{\xi 2}\left(-\xi_1\xi_2^2/r^2+\frac{1}{2}(1-3x_1^2)(1\right. \\ & \left. -\xi_2^2/r^2)\xi_1-\delta\xi_2(1-x_2^2/r^2)\right), \end{aligned} \tag{34}$$

and

$$\Sigma \triangleq \frac{\partial I\!H}{\partial u}=\Lambda_{x2}. \tag{35}$$

On a singular arc, the control is not determined by the necessary condition

$$\frac{\partial I\!H}{\partial u}=\Lambda_{x2}=0. \tag{36}$$

However nonexistence of a singular control for this problem is discussed in detail in reference (Lee, 1991).

Adams' variable-order, variable-step integration method (Shampine, 1975) is employed to integrate (25) and (27), with the local error controlled to less than 1×10^{-9}, from $t=t_0$ to arbitrary chosen t_f with an optimal control computed by (33) and given initial values of $X(t_0)$ and $\Lambda(t_0)$. However, one component of $\Lambda(t_0)$ is computed so that (29) is satisfied at $t=t_0$.

We do not try to solve a two-point boundary-value problem, i. e., a final transversality condition is not satisfied. The reason is, by definition of chaos, that a chaotic system is extremely sensitive to the initial conditions. If we were able to solve a two-point boundary-value problem for a chaotic system, it wouldn't be chaos any more. Also we do not try to integrate the problem backward in time from a point satisfying the terminal transversality condition, because an attractor forward in time becomes a repeller backward in time unless a trajectory starts exactly on the attractor. Backward in time, the state variables diverge from the attractor rapidly. Rather, we try a few different initial starting points which merely satisfy $I\!H(t_0)=0$. Such trajectories are plotted in Fig. 2.

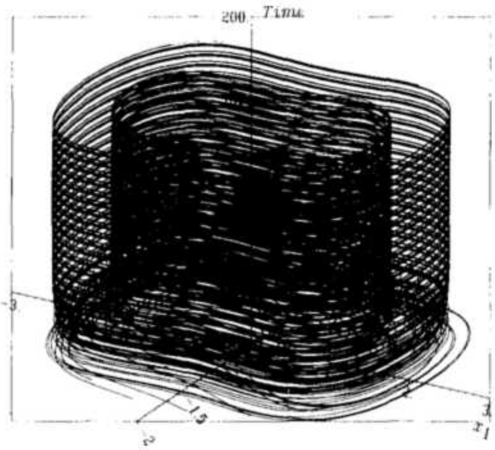

Fig. 2 Trajectories from different starting points

Trajectories are plotted in (x_1, x_2, t) space to visualize behavior more clearly. In Fig. 2, we see that trajectories seem to converge to two different limit cycles, where the inner one has a positive largest Lyapunov exponent while the other has a negative largest Lyapunov exponent. A typical result which has a positive largest Lyapunov exponent is shown in Figs. 3, 4, and 5 with initial conditions :

$$X(t_0)=\begin{Bmatrix}0\\0\\0\\1\\0\end{Bmatrix},\quad \Lambda(t_0)=\begin{Bmatrix}0.7719380248\\0.2274341570\\0.3411512355\\0.4548683140\\-0.1705756177\end{Bmatrix}$$

The rather precise looking numbers above are due to a normalization process for rough initial values.

The results shown in Figs. 3 and 5 are indeed

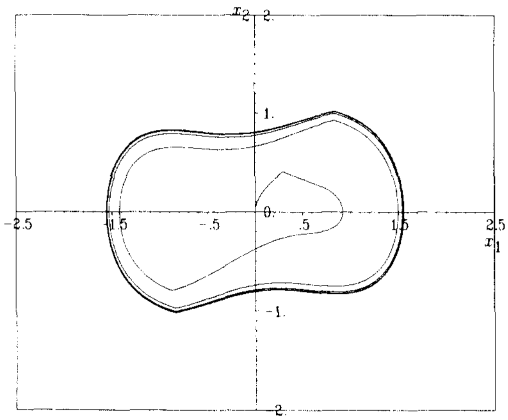

Fig. 3 An optimal trajectory phase portrait

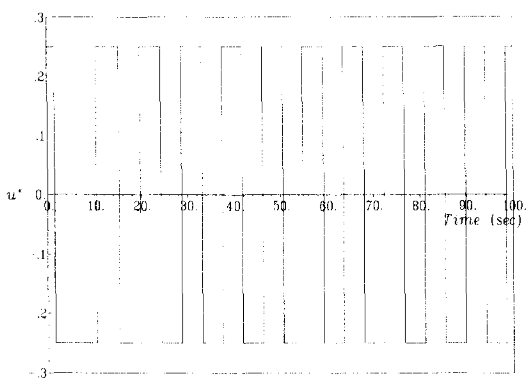

Fig. 4 Optimal control u*(t)

strange. Nearby trajectories are attracted to the limit cycle and the largest Lyapunov exponent is positive, indicating chaotic motion, and is about twice as large as that obtainable with simple sinusoidal forcing at the same amplitude. In addition, the Lyapunov fractal dimension calculated by the conjecture in (Kaplan, 1979) has a noninteger value of approximately 2.5, indicating a fractal strange attractor. However, the "strange" attractor, shown in Fig. 3, is not chaotic and, in fact, is nothing more than a limit cycle. This paradoxical result contradicts the basic idea that a bounded trajectory with at least one positive Lyapunov exponent must be chaotic.

In the following section, it is provided that details of this chaotic limit cycle paradox and the resolution of the paradox. In particular, for systems of differential equations with only piecewise

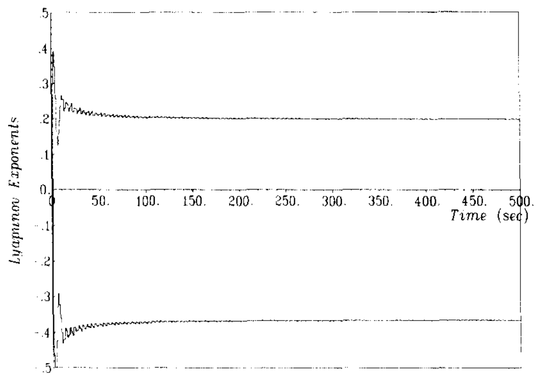

Fig. 5 History of Lyapunov exponents

differentiable right-hand sides and with a closed-loop control, a jump discontinuity condition must be imposed on the state perturbations in order to compute correct Lyapunov exponents.

7. Jump Condition

To discuss the necessary jump condition, let us consider a control law which generates the limit cycle in Fig. 3. The control corresponds to a discontinuous feedback control of the form

$$u(\boldsymbol{x}) = u_{\max}\operatorname{sgn}(m(\boldsymbol{x})), \tag{37}$$

with switching function

$$m(\boldsymbol{x}) = x_2 - kx_1, \ k = 1.1937. \tag{38}$$

We compute the largest Lyapunov exponent for Duffing system (20) with a control in (37) and (38) using the method in (Wolf, 1985) and plot the time history of it in Fig. 6. The feedback control (37) and (38) yields the same value of the largest Lyapunov exponent for a limit cycle as the optimal control does. We see that the method in (Wolf, 1985) for calculating Lyapunov exponents is incorrect for differential equations with discontinuous or nondifferentiable right-hand sides. To use the method in (Wolf, 1985), one must incorporate a jump discontinuity in the state perturbations each time a trajectory crosses the switching surface $m(\boldsymbol{x}) = 0$.

Given a control function $\boldsymbol{u}(\boldsymbol{x}(t))$ at $\boldsymbol{x}(t)$ along the trajectory, Eq. (1) is rewritten as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}(\boldsymbol{x}) \triangleq \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})). \tag{39}$$

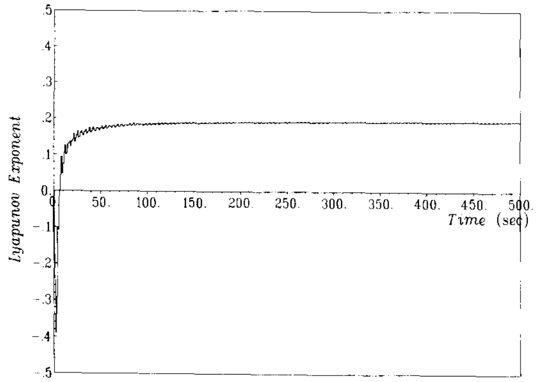

Fig. 6 The largest Lyapunov exponent for duffing system with feedback control

Suppose there is a switching surface which divides the state space in such a way that at the surface the right hand side of (39) is discontinuous. If the path, $\boldsymbol{x}(t)$, generated by (39) passes through the surface then there are jump changes in the state perturbation variables at a point where $\boldsymbol{x}(t)$ crosses the surface (Grantham and Lee, 1993).

Suppose that X is a domain in the state space and is composed of partitions of X_i such that

$$X_i \cap X_j = \emptyset,\ i \neq j$$
$$X_k \subseteq X\ \ k=1,\ 2,\ \cdots,\ K$$

and that an $(n-1)$-dimensional nonempty discontinuity manifold $\mathbb{M}_{ij} \triangleq \bar{X}_i \cap \bar{X}_j,\ i \neq j$ is prespresented by

$$m(\boldsymbol{x})=0$$

in such a way that m is continuous and continuously differentiable on a domain containing $\mathbb{M}_{ij}$ and the function $u(\cdot)$ is defined on each X_k.

Now consider a trajectory which crosses $\mathbb{M}_{ij}$ at point $\boldsymbol{x}_c \triangleq \boldsymbol{x}(\tau_c)$. It is assumed that the trajectory is continuous and that velocity vectors are not tangent to the manifold. Also it is assumed that there exists an $\varepsilon>0$ such that

$$\boldsymbol{x}(\tau) \in X_i\ \ \forall \tau \in (\tau_c-\varepsilon,\ \tau_c)$$
$$\boldsymbol{x}(\tau) \in X_j\ \ \forall \tau \in (\tau_c,\ \tau_c+\varepsilon)$$
$$\frac{\partial m(\boldsymbol{x}_c)}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}_c,\ \boldsymbol{u}_i) \neq 0,$$
$$\boldsymbol{u}_i \triangleq \boldsymbol{u}(\cdot) \text{ at } \tau=\tau_c-\varepsilon$$
$$\frac{\partial m(\boldsymbol{x}_c)}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}_c,\ \boldsymbol{u}_j) \neq 0,\ \ \boldsymbol{u}_j \triangleq \boldsymbol{u}(\cdot)$$
$$\text{at } \tau=\tau_c+\varepsilon. \tag{40}$$

Letting

$$\boldsymbol{f}_- \triangleq \lim_{\substack{\tau \to \tau_c \\ \tau < \tau_c}} \boldsymbol{f}(\boldsymbol{x}(\tau),\ \boldsymbol{u}(\tau))$$
$$\boldsymbol{f}_+ \triangleq \lim_{\substack{\tau \to \tau_c \\ \tau > \tau_c}} \boldsymbol{f}(\boldsymbol{x}(\tau),\ \boldsymbol{u}(\tau))$$
$$\boldsymbol{\eta}_- \triangleq \boldsymbol{\eta}(\tau_c)$$
$$\boldsymbol{\eta}_+ \triangleq \boldsymbol{\eta}(\tau_c+\delta\tau),$$

Grantham and Lee (1993) deduced the following condition at τ_c for $\delta\tau$ and sufficiently small ε:

$$\varepsilon\boldsymbol{\eta}_+ - \boldsymbol{f}_+\delta\tau = \varepsilon\boldsymbol{\eta}_- - \boldsymbol{f}_-\delta\tau. \tag{41}$$

Also, the perturbed state $\bar{\boldsymbol{x}}$ at $\tau_c-\delta\tau$ satisfies $m(\bar{\boldsymbol{x}}(\tau_c-\delta\tau))=0$. Applying the first-order approximation theorem of Taylor, we have

$$\begin{aligned} 0 &= m(\bar{\boldsymbol{x}}(\tau_c-\delta\tau)) \\ &= m(\boldsymbol{x}(\tau_c)-\boldsymbol{f}_-\delta\tau+\varepsilon\boldsymbol{\eta}+\cdots) \\ &= m(\boldsymbol{x})+\frac{\partial m(\boldsymbol{x})}{\partial \boldsymbol{x}}(\varepsilon\boldsymbol{\eta}_- - \boldsymbol{f}_-\delta\tau+\cdots)+\cdots \end{aligned} \tag{42}$$

yielding

$$\frac{\delta\tau}{\varepsilon}=\frac{\frac{\partial m}{\partial \boldsymbol{x}}\boldsymbol{\eta}_-}{\frac{\partial m}{\partial \boldsymbol{x}}\boldsymbol{f}_-}. \tag{43}$$

From (41), we have the following necessary condition for the state perturbation variables at $\boldsymbol{x}(\tau_c)$

$$\boldsymbol{\eta}_+ = \boldsymbol{\eta}_- + (\boldsymbol{f}_+ - \boldsymbol{f}_-)\frac{\delta\tau}{\varepsilon}. \tag{44}$$

Eqs. (43) and (44) constitute a jump condition that should be applied to the state perturbations

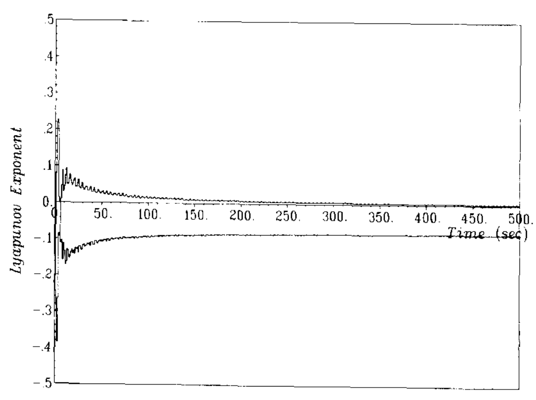

Fig. 7 Lyapunov exponents for duffing system with the jump condition

when the trajectory crosses the switching surface $m(\boldsymbol{x})=0$.

Now consider the Duffing system (20) with the synthesized state feedback control (37) and (38) with the constraint (32). For the system, Lyapunov exponents are computed and plotted in Fig. 7 using the method in (Wolf, 1985) with the jump condition (43) and (44) implemented when the trajectory crosses the switching surface defined by (38). Lyapunov exponents have the correct value for a limit cycle (the largest Lyapunov exponent is zero). The state portrait is not plotted since the Duffing system (20) with a state feedback control (37) and (38) with constraint (32) produces the same limit cycle as in Fig. 3.

8. Open-Loop Optimal Control Problem Solved

The optimal open-loop control indeed yielded the large Lyapunov exponent, given the control constraints. But the resulting trajectory was a limit cycle. In other words, the optimal control yielded a trajectory which has a positive largest Lyapunov exponent and the trajectory was not an strange attractor. The trajectory is chaotic in the sense that nearby trajectories are attracted to it and the trajectory remains in a bounded region with the positive Lyapunov exponent, but the trajectory is not strange in the sense that it is not a space-filling attractor.

The limit cycle was an attractor in the optimal control setting. The optimal control, $u^*(t)$ drove nearby trajectories to the limit cycle. However, when we use the same control to the perturbed trajectory, the perturbed trajectory is not attracted to the limit cycle. Rather, it diverges from the limit cycle. It implies that even if the optimal control is considered as a open-loop control, i. e., a function of time, it act like a closed-loop control which is function of either the state variables or the adjoint variables.

To test the stability, we applied the optimal control as a forcing function to Duffing's equation with an infinitesimally perturbed initial point (perturbed from the initial point which was used in the optimal control problem). The perturbed trajectory soon diverged from the limit cycle shown in Fig. 3, then converged to another limit cycle having a zero largest Lyapunov exponent computed by the method in (Wolf, 1985). A phase portrait of the perturbed trajectory and the first Lyapunov exponent history are shown in Figs. 8 and 9, respectively. In Fig. 9, we can notice that a sharp transition in the value of Lyapunov exponent when the perturbed trajectory breaks out of the unstable limit cycle at about $t=130$ seconds.

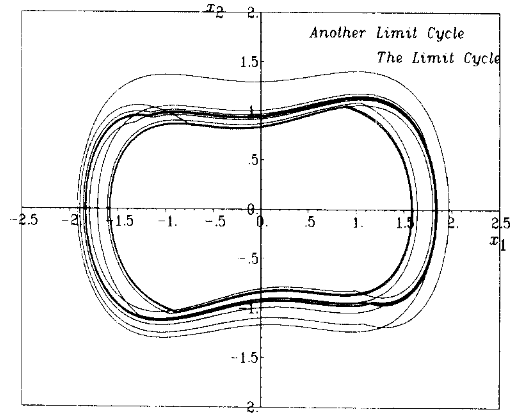

Fig. 8 Diverging nearby trajectory about the chaotic limit cycle

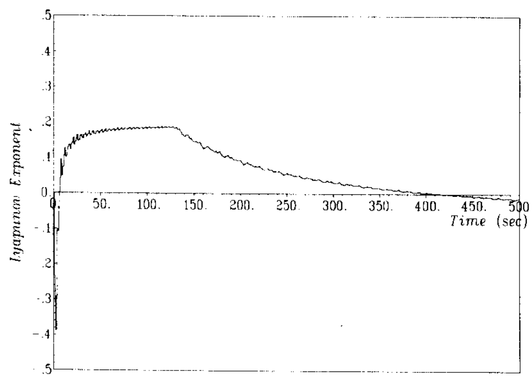

Fig. 9 The first Lyapunov exponent of a perturbed trajectory

9. Closed-Loop Optimal Control Jump Condition in an Augmented Space

In previous sections, we presented nonlinear dynamical systems subject to a Lyapunov exponent maximizing control based on the state perturbations leading to a paradoxical result. The performance index in (13) makes the controller believe it is the most chaotic response that can be achieved, but the optimal control drives trajectories into just an unstable limit cycle. We need to implement a jump discontinuity condition on a perturbation vector for correct Lyapunov exponent computation. In this chapter, we treat an optimal control u^* as a closed-loop control, not in the state space, but in the augmented $(\boldsymbol{x}, \boldsymbol{\lambda})$ space. For example, in Duffing's equation, u^* is a bang-bang control whose switching function is λ_{x_2}. In this case u^* is a function of one of the adjoint variables. Bang-bang control occurs, for example, if u appears linearly in the function and the performance integrand is explicitly independent of u. We can find many problems which fall in this category.

An optimal control u^* satisfying condition (28) can be rewritten as

$$\boldsymbol{u}^* = \underset{\boldsymbol{u}}{\text{argmax}}\, I\!H(\Lambda(t), \boldsymbol{X}^*(t), \boldsymbol{u}). \tag{45}$$

With closed-loop control $u(\boldsymbol{x}, \boldsymbol{\lambda})$in mind, we can convert Eqs. (25)~(29) into a problem in the augmented space given by

$$\dot{\chi} = \mathrm{F}[\chi, u(\chi)] \tag{46}$$

with a closed-loop control

$$u(\chi) = \underset{u}{\text{argmax}}\, I\!H(\chi, u) \tag{47}$$

subject to a control constraint $|u| \leq u_{\max}$, with a switching surface

$$M(\chi) = 0 \tag{48}$$

across which control u is discontinuous and an initial constraint, such as $I\!H(\cdot) = 0$, of the form

$$G(\chi) = 0, \tag{49}$$

where

$$\chi \triangleq \begin{Bmatrix} \boldsymbol{X} \\ \boldsymbol{\Lambda} \end{Bmatrix}, \quad F \triangleq \begin{Bmatrix} \dot{\boldsymbol{X}} \\ \dot{\boldsymbol{\Lambda}} \end{Bmatrix}.$$

To compute a Lyapunov exponent for (46) with a closed-loop control (47) and the initial condition (49), let us consider a perturbation vector, $\Delta\boldsymbol{\chi}$, with a governing differential equation given by

$$\frac{d(\Delta\chi)}{dt} = \left(\frac{\partial F}{\partial \chi} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial \chi}\right)\Delta\chi \tag{50}$$

and satisfying the initial condition (49) at the perturbed point, so that

$$G(\chi + \Delta\chi) = 0 \tag{51}$$

Applying a jump condition, we obtain

$$\Delta\chi_+ = \Delta\chi_- + (F_+ - F_-)\frac{\dfrac{\partial M}{\partial \chi}\cdot\Delta\chi_-}{\dfrac{\partial M}{\partial \chi}\cdot F_-} \tag{52}$$

which should be applied when u switches across the switching surface (48), with an initial value of $\Delta\chi$:

Note that the choice of initial $\Delta\chi$ is not totally arbitrary since not only the closed-loop control problem (47)~(50) is dependent on the optimal problem in (14) but also the perturbed point should satisfy (51). However, we still have freedom left for choosing the initial $\Delta\chi$

$$\Delta\chi_0 \triangleq \begin{Bmatrix} \Delta x_0 \\ \Delta\boldsymbol{x}_0 \\ \Delta\boldsymbol{\xi}_0 \\ \Delta\lambda_0 \\ \Delta\boldsymbol{\lambda}_{x_0} \\ \Delta\boldsymbol{\lambda}_{\xi_0} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boldsymbol{\xi} \\ \boldsymbol{0} \\ 0 \\ \Delta\boldsymbol{\lambda}_{x_0} \\ \boldsymbol{0} \end{Bmatrix} \tag{53}$$

which yields the correct Lyapunov exponent. In the following section, we will consider a system which has a known optimal cost-to-go function so that the initial perturbation vector is well defined.

10. An Example having Known Optimal Cost Return Function

Consider a system, developed in (Leitmann, 1981), having a unit mass in rectilinear motion subject to a force, given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t), \end{aligned} \tag{54}$$

with the control constrained by

$$|\mathrm{u}(t)| \leq 1. \tag{55}$$

The objective of the problem is to transfer the state from a given initial point, $\boldsymbol{x}_0$, to the origin in minimum time. Thus we have $f_0 = 1$.

To utilize the maximum principle, we have the function H defined by

$$H(\lambda, x, u) = \lambda_0 + \lambda_1 x_2 + \lambda_2 u \tag{56}$$

with $\lambda_0 = 1$ and the adjoint equations

$$\dot{\lambda}_1(t) = 0$$
$$\dot{\lambda}_2(t) = -\lambda_1(t). \tag{57}$$

Thus we obtain

$$\lambda_1(t) = \lambda_1(t_f)$$
$$\lambda_2(t) = \lambda_2(t_f) + \lambda_1(t_f)(t_f - t). \tag{58}$$

Letting $c_1 \triangleq \lambda_1(t_f)$ and $c_2 \triangleq \lambda_1(t_f)t_f + \lambda_2(t_f)$ we have

$$H[\lambda(t), x(t), u] = \lambda_0 + c_1 x_2(t) + (c_2 - c_1 t)u. \tag{59}$$

The switching function $\Sigma \triangleq \partial H/\partial u$ is given by

$$\Sigma(t) = \lambda_2(t) = c_2 - c_1 t \tag{60}$$

where c_1 and c_2 both cannot be zero, implying that the extremal control, $u(\cdot)$is bang-bang control with at most one switch.

From the development in (Leitmann, 1981) of the synthesized optimal control, we have u^* defined by

$$u^* = \begin{cases} 0 \text{ if } x_1 = x_2 = 0 \\ -1 \text{ if } (x_2 > 0 \text{ and } 2x_1 + x_2^2 \geq 0) \\ \qquad \text{or } 2x_1 - x_2^2 > 0 \\ 1 \text{ if } (x_2 < 0 \text{ and } 2x_1 + x_2^2 \leq 0) \\ \qquad \text{or } 2x_1 + x_2^2 < 0 \end{cases} \tag{61}$$

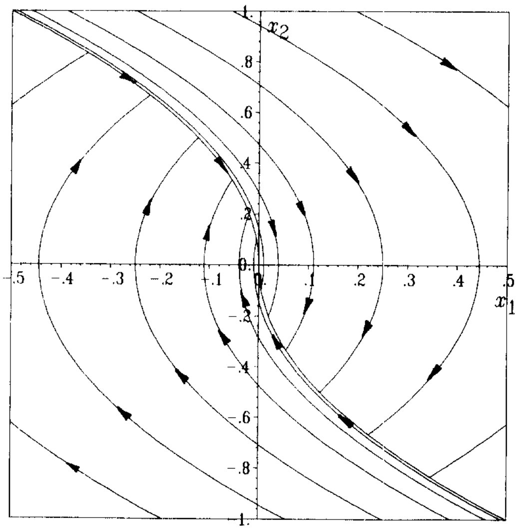

Fig. 10 Time optimal trajectories

Using the optimal control law in (61) for (54), optimal trajectories are plotted in Fig. 10. Also, the optimal cost return function defined by

$$V^*(x_0) \triangleq \int_{t_0}^{t_f} f_0 dt \tag{62}$$

has the property that

$$\lambda^T(t) = \frac{\partial V^*(x^*(t))}{\partial x}, \tag{63}$$

which lets us determine the value of the initial perturbation vector $\Delta\chi$. For the example problem,

$$V^*(x) = \begin{cases} x_2 + \sqrt{4x_1 + 2x_2^2} & \text{for } x_1 > -\frac{1}{2}x_2|x_2|, \\ -x_2 + \sqrt{-4x_1 + 2x_2^2} & \text{for } x_1 < -\frac{1}{2}x_2|x_2|, \\ |x_2| & \text{for } x_1 = -\frac{1}{2}x_2|x_2|. \end{cases} \tag{64}$$

To determine initial perturbations, $\Delta\lambda_x$ in (53), we use the variational equation of (63) as follows:

$$\Delta\lambda_x = \frac{\partial^2 V^*}{\partial x^2}\Delta x \tag{65}$$

to the first order.

We applied the necessary condition (52) to the example problem knowing the switching function (60) and the optimal cost return function. To do so, we chose an arbitrary initial point in the state space and corresponding adjoint variables were determined using (63). Then, for arbitrary small state perturbation vector ξ in (53), the perturbation in the adjoint vector $\Delta\lambda_x$ was determined according to (65). With given initial values, the governing differential equations were integrated with the control sequences given by (61) and the jump condition (52) was applied at the switching time. The plots for reference and perturbation trajectories with the jump condition implemented are shown in Fig. 11. In the figure, at switching time, we see that the perturbed trajectory was corrected by the jump condition with a negligible error as though the closed-loop control law (61)

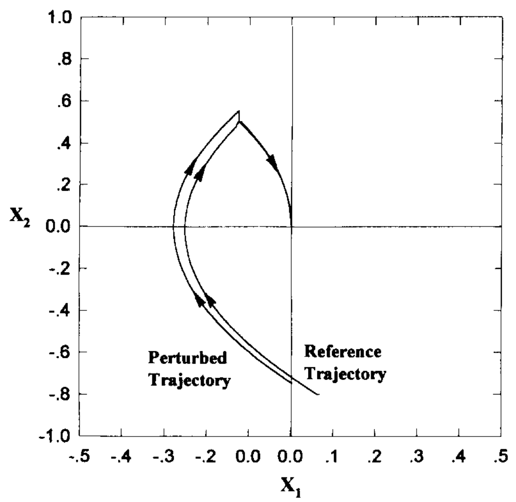

Fig. 11 Reference and perturbed trajectories with jump condition implemented

has been applied to the perturbed trajectory.

11. Summary

An optimal control problem has been formulated to determine the most chaotic response achievable for dynamical systems by manipulating the largest Lyapunov exponent as a chaos measurement. However, the application of current optimal control theory for maximizing the largest Lyapunov exponent presents difficulties. The difficulties are exponential growth of the norm of the state perturbation vector and the norm of the adjoint variables. A common approach for avoiding this computational problem is periodic renormalization. However, periodic renormalization raises a discontinuity problem which is not a standard case in optimal control theory. To circumvent the exponential growth in magnitude and the discontinuity problem, we employ a method of continuous normalization which replaces periodic discontinuous renormalization with differential equations that correspond to continuous normalization at each instant of time.

Next, an optimal control problem was formulated and applied to the Duffing's equation to maximize the largest Lyapunov exponent. The resulting open-loop optimal controller yielded a paradoxical result that an attractor having a positive largest Lyapunov exponent was not a chaotic strange attractor. Indeed, the resulting phase portrait of the optimal trajectory was a limit cycle. The conclusion is that even if the optimal control is considered as an open-loop control, i.e., a function of time, it acts like a closed-loop control, i.e., a function of the state and the adjoint variables. For an optimal control as a closed-loop control we need to implement a jump discontinuity condition on a perturbation vector for correct Lyapunov exponent computation. This will eventually resolve the chaotic limit cycle paradox.

For the present, however, we have only demonstrated the optimal closed-loop jump condition for a system in which the optimal return function was known, so that we could directly calculate initial adjoint perturbations as function of initial state perturbations.

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